STABILITY OF FLUID FILM ON THE SURFACE OF A ROTATING CYLINDER

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The stability of a viscous fluid film retained by surface tension forces on the surface of an infinite rotating cylinder is examined. It is shown that the cylindrical configuration of the fluid film is unstable even for arbitrarily slow rotations as compared to the long wave cyclic axisymmetric disturbances.

In the undisturbed motion of the film, only the azimuthal velocity component is other than zero (stable rotation),

$$V_\varphi = \Omega r, \qquad P = \frac{\rho\Omega^2 r^2}{2} - \frac{\rho\Omega^2 (R+h)^2}{2} + \frac{\alpha}{R+h}. \tag{1}$$

Here, Ω is the angular velocity of the cylinder (directed along the z-axis of the cylinder), P is the pressure, R is the radius of the solid cylinder, h is the film thickness, and α is the surface tension coefficient.

Let the disturbances of both the free surface of the film $hf(z, t)$ and of the flow be axisymmetrical,

$$\begin{gathered} v_r = u(r)\, e^{ikz+\sigma t}, \qquad V_\varphi + v(r)\, e^{ikz+\sigma t}, \\ v_z = w(r)\, e^{ikz+\sigma t}, \qquad p = p(r)\, e^{ikz+\sigma t}, \\ R + h + hf(z, t) = R + h + h\varepsilon e^{ikz+\sigma t}, \\ \partial(\ldots)/\partial\varphi \equiv 0, \qquad \varepsilon \ll 1. \end{gathered} \tag{2}$$

We examine the Navier-Stokes and continuity equations linearized with respect to the disturbance:

$$\begin{gathered} \sigma u - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{d^2u}{dr^2} - k^2 u + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right], \\ \sigma v + 2\Omega u = \nu\left[\frac{d^2v}{dr^2} - k^2 v + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right], \\ \sigma w = -\frac{ik}{\rho} p + \nu\left[\frac{d^2w}{dr^2} - k^2 w + \frac{1}{r}\frac{dw}{dr}\right], \end{gathered} \tag{3}$$

$$\frac{du}{dr} + \frac{u}{r} + ikw = 0. \tag{4}$$

We have the following boundary conditions:
for r = R,

$$u = v = W = 0; \tag{5}$$

at the free surface,

$$\sigma_{ik} n_k = -\alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right) n_i, \qquad v_r = \frac{d}{dt}(hf) = \frac{\partial (hf)}{\partial t}. \tag{6}$$

The stress tensor σ_{ik} and the unit vector n_i are taken in a cylindrical system of coordinates [1].

Here, $n_\varphi = 0$ by virtue of the axial symmetry of the disturbance, while $n_z = h df/dz$ is on the order of the level of disturbance. The expression for the principal radii of curvature is also linearized with respect to the disturbance,

$$R_1 = R + h + hf(z, t), \qquad R_2^{-1} = -h\frac{\partial^2 f}{\partial z^2}. \tag{7}$$

Considering this, we get the boundary conditions at the free surface in the form

$$\begin{gathered} -\Omega^2 (R+h) h\varepsilon - \frac{p}{\rho} + 2\nu\frac{du}{dr} = \frac{\alpha}{\rho}\left(\frac{h\varepsilon}{(R+h)^2} - hk^2\varepsilon\right), \\ \frac{dv}{dr} - \frac{v}{r} = 0, \qquad \frac{dw}{dr} + iku = 0, \qquad u = \sigma h\varepsilon. \end{gathered} \tag{8}$$

In the case of a thin film, we set $h/R \ll 1$, and introduce a new variable,

$$r = R + hx. \tag{9}$$

Then,

$$\frac{d}{dr} = \frac{1}{h}\frac{d}{dx}, \qquad \frac{1}{r} = \frac{1}{R} + O\left(\frac{h}{R}\right). \tag{10}$$

Treating h/R as a small parameter, we expand (3)-(8) in series in this parameter. After eliminating the pressure p and w(r), in zero approximation we have

$$\begin{gathered} \left[\frac{d^2}{dx^2} - \left(\frac{\sigma h^2}{\nu} + k^2h^2\right)\right]\left[\frac{d^2}{dx^2} - k^2h^2\right] u = \frac{2\,\Omega k^2 h^4}{\nu} v, \\ \left[\frac{d^2}{dx^2} - \left(\frac{\sigma h^2}{\nu} + k^2h^2\right)\right] v = \frac{2\,\Omega h^2}{\nu} u, \\ u = v = \frac{du}{dx} = 0 \quad \text{for } x = 0, \\ x = 1, \quad \left[\frac{d^2}{dx^2} - \left(\frac{\sigma h^2}{\nu} + k^2h^2\right)\right]\frac{du}{dx} - \\ - 2k^2h^2\frac{du}{dx} + \frac{k^2h^3}{\sigma\nu}\left(\Omega^2 R - \frac{\alpha}{\rho} k^2\right) u = 0, \\ \frac{d^2u}{dx^2} + k^2h^2 u = 0, \qquad \frac{dv}{dx} = 0. \end{gathered} \tag{11}$$

Let us examine longwave disturbances. By setting

$$kh \ll 1, \qquad \frac{\sigma h^2}{\nu} \ll 1, \qquad \frac{\Omega h^2}{\nu} \ll 1,$$

expanding the solution in series in the small parameters kh, $\Omega h^2/\nu$, $\sigma h^2/\nu$, and limiting the analysis to the principal terms of the expansion, we get from (11) the following expression:

$$u(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2, \frac{\sigma h^2}{\nu}\right). \tag{12}$$

By substituting u(x) into boundary conditions (11), we get for σ the secular equation

$$\begin{gathered} \det \|a_{ij}\| = 0 \quad (ij = 1, 2), \\ a_{11} = \frac{k^2h^3}{\sigma\nu}\left(\Omega^2 R - \frac{\alpha}{\rho} k^2\right) + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2\right), \\ a_{21} = 2 + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2\right), \\ a_{12} = \frac{k^2h^3}{\sigma\nu}\left(\Omega^2 R - \frac{\alpha}{\rho} k^2\right) + 6 + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2\right), \\ a_{22} = 6 + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2\right). \end{gathered} \tag{13}$$

Hence,

$$\sigma = \frac{k^2h^3}{3\nu}\left(\Omega^2 R - \frac{\alpha}{\rho} k^2\right)\left[1 + O\left(\frac{\Omega^2 h^4}{\nu^2}, k^2h^2, \frac{\sigma h^2}{\nu}\right)\right]. \tag{14}$$

Thus, a cylindrical film is unstable with respect to disturbances with wave numbers

$$k < \left(\frac{\rho\Omega^2 R}{\alpha}\right)^{1/2}.$$

REFERENCE

1. L. D. Landau and E. M. Lifshitz, Mechanics of Continuous Media [in Russian], Gostekhizdat, Moscow, 1953.

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